

## Products of Unipotent Matrices of Index 2

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### ABSTRACT

We show that an  $n \times n$  complex matrix  $T$  is the product of two unipotent matrices of index 2 if and only if  $T$  is similar to a matrix of the form  $D \oplus D^{-1} \oplus (I + N) \oplus (-I + \sum_{i=1}^m \oplus J_i)$ , where 0 and  $\pm 1$  are not eigenvalues of  $D$ ,  $N$  is nilpotent, and each  $J_i$  is a nilpotent Jordan block of even size. On the other hand,  $T$  is the product of finitely many unipotent matrices of index 2 if and only if  $\det T = 1$ . In this case, the minimal number of required unipotents is 1 if  $n = 1$ , 3 if  $n = 2$ , and 4 if  $n \geq 3$ .

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### 1. INTRODUCTION

An  $n \times n$  complex matrix  $U$  is *unipotent* if  $U = I + N$ , where  $I$  is the identity matrix and  $N$  is nilpotent. It is *unipotent of index  $m$*  if  $N^m = 0$  and  $N^{m-1} \neq 0$ . Fong and Sourour [4] initiated the study of the factorization of complex matrices into unipotent ones. They showed that every complex matrix  $T$  with determinant 1 is the product of three unipotents. Using the factorization theorem in [11], we can easily show that  $T$  is the product of two unipotents if and only if either  $T$  is the identity matrix or  $T$  is nonscalar with determinant 1.

In this paper, we restrict ourselves to unipotent matrices with index 2 and consider the problems of characterizing matrices which are expressible as products of two or more such matrices. We start in Section 2 by studying

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products of two unipotent matrices of index 2. We are able to completely characterize this class (Theorem 2.7). Specifically, we show that a complex matrix  $T$  is the product of two unipotent matrices of index 2 if and only if  $T$  is similar to a matrix of the form  $D \oplus D^{-1} \oplus (I + N) \oplus (-I + \sum_{i=1}^m 1 \oplus J_i)$ , where 0 and  $\pm 1$  are not eigenvalues of  $D$ ,  $N$  is nilpotent, and each  $J_i$  is a nilpotent Jordan block of even size.

In Section 3, we consider the problem of expressing matrices as products of three or more unipotents of index 2. We show that every complex matrix with determinant 1 is the product of four unipotent matrices of index 2 (Theorem 3.5) and proceed to determine whether fewer of them will do. Depending on the size of the matrices considered, the minimal such number can be completely determined. This is achieved through an examination on matrices which are expressible as products of three unipotents of index 2. Although we haven't been able to give a complete characterization of such matrices, we do obtain some necessary or sufficient conditions. In particular, we show that if the  $n \times n$  matrix  $T$  is the product of three unipotents of index 2, then the geometric multiplicity of any eigenvalue, other than  $\pm 1$ , of  $T$  is at most three-fourths of  $n$  (Theorem 3.1).

We conclude this introduction by bringing attention to the close resemblance between the theory of products of unipotent matrices of index 2 and that of products of involutions (cf. Corollary 2.8, and [1], [2], [3], [5], [8], [9], and [13]). These multiplicative theories are also parallel to the additive ones of sums of idempotent and square-zero matrices (cf. [6], [7], [12], and [14]).

For the rest of the paper, all unipotent matrices are assumed to be of index 2.

## 2. TWO UNIPOTENTS

For a matrix  $T$ ,  $\sigma(T)$  denotes the set of its eigenvalues. We start with the following

**LEMMA 2.1.** *Let  $T = T_1 \oplus T_2$  be an invertible matrix with  $\sigma(T_1) \cap \sigma(T_2^{-1}) = \emptyset$ . Then  $T$  is the product of two unipotents if and only if both  $T_1$  and  $T_2$  are.*

*Proof.* We need only prove the necessity part. Assume that  $T = (I + S)(I + R)$ , where

$$S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}$$

are square-zero matrices, that is,  $S^2 = R^2 = 0$ . A little computation yields that

$$\begin{aligned} S_1^2 + S_2S_3 &= 0, & R_1^2 + R_2R_3 &= 0, \\ S_1S_2 + S_2S_4 &= 0, & R_1R_2 + R_2R_4 &= 0, \\ S_3S_1 + S_4S_3 &= 0, & R_3R_1 + R_4R_3 &= 0, \\ S_3S_2 + S_4^2 &= 0, & R_3R_2 + R_4^2 &= 0 \end{aligned}$$

and

$$\begin{aligned} (I + S_1)(I + R_1) + S_2R_3 &= T_1, \\ (I + S_1)R_2 + S_2(I + R_4) &= 0, \\ S_3(I + R_1) + (I + S_4)R_3 &= 0, \\ S_3R_2 + (I + S_4)(I + R_4) &= T_2. \end{aligned}$$

Hence

$$\begin{aligned} S_2T_2 &= S_2S_3R_2 + S_2(I + S_4)(I + R_4) \\ &= -S_1^2R_2 + (S_2 - S_1S_2)(I + R_4) \\ &= -S_1^2R_2 + (I - S_1)S_2(I + R_4) \\ &= -S_1^2R_2 - (I - S_1^2)R_2 \\ &= -R_2. \end{aligned}$$

Similarly, we have  $T_1R_2 = -S_2$ ,  $S_3T_1 = -R_3$ , and  $T_2R_3 = -S_3$ . Therefore,  $S_2T_2 = T_1^{-1}S_2$  or  $T_1S_2 = T_2^{-1}S_2$ . Thus  $\sigma(T_1) \cap \sigma(T_2^{-1}) = \emptyset$  implies that  $S_2 = 0$  (cf. [10]). By symmetry,  $R_2 = 0$ . So  $T_1 = (I + S_1)(I + R_1)$  is the product of two unipotents. The same holds for  $T_2$ . ■

The next lemma gives a necessary condition for products of two unipotents.

**LEMMA 2.2.** *If  $T$  is the product of two unipotents, then  $T$  is similar to  $T^{-1}$ .*

*Proof.* Since  $T$  is similar to  $T_1 \oplus T_2$ , where  $1 \notin \sigma(T_1)$  and  $\sigma(T_2) = \{1\}$ , Lemma 2.1 implies that  $T_1$  is a product of two unipotents. Say  $T_1 = (I + S)(I + R)$ , where  $S^2 = R^2 = 0$ . Then  $T_1^{-1} = (I - R)(I - S)$ , and therefore  $(T_1 - I)(T_1^{-1} - I) = 2I - T_1 - T_1^{-1} = -SR - RS = (S - R)^2$ . The invertibility of  $T_1 - I$  implies that of  $S - R$ . Now, since

$$\begin{aligned} (S - R)T_1 &= (S - R)(I + S + R + SR) \\ &= S + SR - R - RS - RSR \\ &= (I - S - R + RS)(S - R) \\ &= T_1^{-1}(S - R), \end{aligned}$$

the similarity of  $T_1$  and  $T_1^{-1}$  follows. That  $T_2$  is similar to  $T_2^{-1}$  is a consequence of [3, Lemma 1], and thus the same holds for  $T$ . ■

Note that the converse of Lemma 2.2 is not true in general. In fact, it is known that if  $\sigma(T) = \{-1\}$  then  $T$  is similar to  $T^{-1}$  (cf. [3, Lemma 1]), but not every such  $T$  is the product of two unipotents, as the following lemma shows. Recall that  $J_k$  denotes the nilpotent Jordan block of size  $k$ :

$$J_k = \begin{pmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}.$$

LEMMA 2.3. *If  $T = (-I + J_k) \oplus \sum_{i=1}^m (-I + J_{k_i})$ , where  $k$  is odd, then  $T$  is not the product of two unipotents.*

*Proof.* Suppose that  $T$  is a product of two unipotents:  $T = (I + S)(I + R)$ . We obtain, as in the proof of Lemma 2.1 with  $T_1 = -I + J_k$  and  $T_2 = \sum_{i=1}^m (-I + J_{k_i})$ , that

$$\begin{aligned} S_2 S_3 &= -S_2 T_2 R_3 = R_2 R_3, \\ S_2 S_3 T_1 &= -S_2 R_3 = T_1 R_2 R_3 = T_1 S_2 S_3 \end{aligned}$$

and

$$R_1 S_2 S_3 = R_1 R_2 R_3 = -R_2 R_4 R_3 = R_2 R_3 R_1 = S_2 S_3 R_1.$$

We consider two cases separately.

*Case 1.  $S_2S_3$  is invertible.* From the equations in the proof of Lemma 2.1 and the above ones, we obtain

$$\begin{aligned} 0 &= S_2[S_3(I + R_1) + (I + S_4)R_3] \\ &= S_2S_3(I + R_1) + (S_2 - S_1S_2)R_3 \\ &= (I + R_1)S_2S_3 + (I - S_1)S_2R_3 \\ &= (I + R_1)S_2S_3 - (I - S_1)T_1S_2S_3. \end{aligned}$$

Therefore,

$$I + R_1 = (I - S_1)T_1. \quad (1)$$

Similarly, we have

$$I + S_1 = T_1(I - R_1). \quad (2)$$

If  $k = 1$ , then  $T_1 = -1$ . Adding (1) and (2) yields the ludicrous  $2 = -2$ . Thus in the following we may assume that  $k > 1$ . Let  $I - S_1$  be the  $k \times k$  matrix  $[x_{ij}]$ . Then (1) and (2) imply that

$$\begin{aligned} 2I - [x_{ij}] &= I + S_1 = T_1(I - R_1) = T_1\{2I - (I + R_1)\} \\ &= (-I + J_k)\{2I - [x_{ij}]\}(-I + J_k). \end{aligned}$$

Carrying out the above multiplication and comparing the entries in the lower triangular parts of the resulting matrices on both sides yields that  $x_{21} = 4$ ,  $x_{31} = \cdots = x_{k1} = 0$ ,  $x_{32} = \cdots = x_{k2} = 0$ ,  $x_{43} = 4$ ,  $x_{53} = \cdots = x_{k3} = 0$ , and so on. In particular, we obtain

$$x_{jj-1} = \begin{cases} 4 & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd} \end{cases}$$

for  $2 \leq j \leq k$ . But comparing the  $(k, k)$  entries, we have  $2 - x_{kk} = -2 - x_{kk} + x_{kk-1}$ , which implies that  $x_{kk-1} = 4$ . This yields a contradiction, since  $k$  is odd.

*Case 2.  $S_2S_3$  is noninvertible.* If  $k = 1$ , then  $S_2S_3 = R_2R_3 = 0$ . From  $S_1^2 + S_2S_3 = 0$  and  $R_1^2 + R_2R_3 = 0$ , we deduce that  $S_1 = R_1 = 0$ . Also,  $S_2R_3 = -S_2S_3T_1 = 0$ . Therefore, from  $(I + S_1)(I + R_1) + S_2R_3 = T_1$ , we obtain  $1 = -1$ , which is ridiculous. Hence for the rest of the proof we may assume that  $k > 1$ . Since, as verified before,  $S_2S_3$  commutes with  $J_k = I + T_1$ , we

have  $S_2 S_3 = \sum_{j=0}^{k-1} a_j J_k^j$  for some complex numbers  $a_0, \dots, a_{k-1}$ . The noninvertibility of  $S_2 S_3$  forces  $a_0 = 0$ . Hence

$$\begin{aligned} \det(T_1 - S_2 R_3) &= \det\{T_1(I + S_2 S_3)\} \\ &= \det\left\{(-I + J_k)\left(I + \sum_{j=1}^{k-1} a_j J_k^j\right)\right\} \\ &= \det\left(-I + (1 - a_1)J_k + \sum_{j=2}^{k-1} (a_{j-1} - a_j)J_k^j\right) = -1, \end{aligned}$$

since  $k$  is odd. On the other hand, by the identities  $S_1^2 = -S_2 S_3$  and  $R_1^2 = -R_2 R_3 = -S_2 S_3$ ,  $S_1$  and  $R_1$  are nilpotent. Therefore,  $\det(I + S_1) = \det(I + R_1) = 1$ . It follows from  $(I + S_1)(I + R_1) + S_2 R_3 = T_1$  that  $1 = \det(I + S_1)\det(I + R_1) = \det(T_1 - S_2 R_3) = -1$ , which is impossible. This completes the proof.  $\blacksquare$

To prove the sufficiency condition of our main theorem (Theorem 2.7 below), we need the following lemma from [4, Lemma 3].

**LEMMA 2.4.** *If  $T$  and  $I + T$  are invertible, then  $T \oplus T^{-1}$  is the product of two unipotents.*

The next two lemmas dig out the matrices  $T$  with  $\sigma(T) = \{\pm 1\}$  which are expressible as products of two unipotents.

**LEMMA 2.5.** *Any matrix  $T$  with  $\sigma(T) = \{1\}$  is the product of two unipotents.*

*Proof.* It suffices to show that for each integer  $k \geq 2$ ,  $I + J_k$  is the product of two unipotents. Let

$$M = \begin{cases} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_m & \text{if } k = 2m, \\ \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_m \oplus [1] & \text{if } k = 2m + 1 \end{cases}$$

and

$$N = \begin{cases} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & \text{if } k = 2, \\ [1] \oplus \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{m-1} \oplus [1] & \text{if } k = 2m \geq 4, \\ [1] \oplus \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_m & \text{if } k = 2m + 1. \end{cases}$$

It is easily seen that  $M$  and  $N$  are unipotent, and

$$MN = \begin{cases} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} & \text{if } k = 2, \\ \begin{bmatrix} 1 & 1 & & * \\ & 1 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 1 \end{bmatrix} & \text{if } k \geq 3. \end{cases}$$

In either case,  $MN$  is similar to  $I + J_k$ . Hence  $I + J_k$  is the product of two unipotents. ■

LEMMA 2.6.  $-I + J_k$  is the product of two unipotents for any even  $k$ .

*Proof.* Let

$$M = \begin{bmatrix} -4 & 4 \\ -4 & 4 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} -6 & 9 \\ -4 & 6 \end{bmatrix},$$

and let  $M_0$  and  $N_3$  be the  $k \times k$  matrices

$$\begin{bmatrix} M & \cdots & M \\ & \ddots & \vdots \\ 0 & & M \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} N & \cdots & N \\ & \ddots & \vdots \\ 0 & & N \end{bmatrix}.$$

respectively. It is easily seen that  $M_0^2 = N_0^2 = 0$  and

$$(I + M_0)(I + N_0)$$

$$= \begin{bmatrix} (I + M)(I + N) & M + N + 2MN & & & & & * \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & & M + N + 2MN \\ & & & & & & (I + M)(I + N) \\ 0 & & & & & & \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 6 & -11 & & & * \\ & -1 & 8 & -14 & & & \\ & & -1 & 1 & 6 & -11 & \\ & & & -1 & 8 & -14 & \\ & & & & \ddots & \ddots & \\ & & & & & -1 & 1 \\ 0 & & & & & & -1 \end{bmatrix}$$

$$= -I + \begin{bmatrix} 0 & 1 & & & & & * \\ & 0 & 8 & & & & \\ & & 0 & 1 & & & \\ & & & 0 & 8 & & \\ & & & & \ddots & \ddots & \\ & & & & & 1 & \\ 0 & & & & & & 0 \end{bmatrix}.$$

Since this latter matrix is similar to  $-I + J_k$ , our assertion follows.  $\blacksquare$

Combining the above lemmas, we obtain

**THEOREM 2.7.** *A matrix  $T$  is the product of two unipotents if and only if  $T$  is similar to  $D \oplus D^{-1} \oplus (I + N) \oplus (-I + \sum_{i=1}^m \oplus J_{k_i})$ , where  $0, \pm 1 \notin \sigma(D)$ ,  $N$  is nilpotent, and  $k_i$  is even for each  $i$ .*

*Proof.* The sufficiency follows from Lemmas 2.4, 2.5, and 2.6. To prove the necessity, note that  $T$  is similar to  $T_1 \oplus T_2 \oplus T_3$ , where  $\pm 1 \notin \sigma(T_1)$ ,  $\sigma(T_2) = \{1\}$ , and  $\sigma(T_3) = \{-1\}$ . Lemmas 2.1 and 2.3 imply that  $T_1$  is the product of two unipotents and  $T_3$  is similar to  $-I + \sum_{i=1}^m \oplus J_{k_i}$ , where  $k_i$  is even for each  $i$ . Thus  $T_1$  is similar to  $T_1^{-1}$  by Lemma 2.2. Since  $T_1$  is similar to  $D_1 \oplus D_2$ , where  $\sigma(D_1) \subseteq \{z : |z| < 1 \text{ or } |z| = 1 \text{ and } \operatorname{Im} z > 0\}$  and  $\sigma(D_2) \subseteq \{z : |z| > 1 \text{ or } |z| = 1 \text{ and } \operatorname{Im} z < 0\}$ , the similarity of  $T_1$  and  $T_1^{-1}$  implies the



existence of an invertible matrix

$$U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}$$

such that

$$\begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} \begin{bmatrix} D_1^{-1} & 0 \\ 0 & D_2^{-1} \end{bmatrix}.$$

Then  $D_1 U_1 = U_1 D_1^{-1}$  and  $\sigma(D_1) \cap \sigma(D_1^{-1}) = \emptyset$  imply that  $U_1 = 0$  (cf. [10]). Similarly,  $U_4 = 0$ . Hence both  $U_2$  and  $U_3$  are invertible, and thus  $D_2$  is similar to  $D_1^{-1}$ , as desired. ■

Recall that a matrix  $T$  is an *involution* if  $T^2 = I$ .

**COROLLARY 2.8.** *If the matrix  $T$  is such that  $T$  and  $I + T$  are invertible, then the following statements are equivalent:*

- (1)  *$T$  is the product of two unipotents;*
- (2)  *$T$  is the product of two involutions;*
- (3)  *$T$  is similar to  $T^{-1}$ ;*
- (4) *there exists an involution  $V$  such that  $TV = VT^{-1}$ ;*
- (5)  *$T$  is similar to  $D \oplus D^{-1} \oplus (I + N)$ , where  $D$  is invertible and  $N$  is nilpotent.*

*Proof.* This is an easy consequence of [3, Theorem 1], [1, Theorem 2], and Theorem 2.7. ■

### 3. THREE OR MORE UNIPOTENTS

In this section, we consider the problem of expressing matrices as products of three or more unipotents. We start with the following necessary condition for products of three unipotents.

**THEOREM 3.1.** *If the  $n \times n$  matrix  $T$  is a product of three unipotents, then  $\dim \ker(T - \alpha I) \leq \frac{3}{4}n$  for any  $\alpha \in \mathbb{C}$ ,  $\alpha^2 \neq 1$ .*

*Proof.* Let  $T = (I + T_1)(I + T_2)(I + T_3)$ , where  $T_j^2 = 0$  for  $j = 1, 2, 3$ . Note that  $\dim \ker T_j \geq \frac{1}{2}n$  for all  $j$ . Indeed, if  $\dim \ker T_j < \frac{1}{2}n$ , then  $\text{ran } T_j \subseteq$

$\ker T_j$  implies that  $\text{rank } T_j < \frac{1}{2}n$ , whence  $n = \dim \ker T_j + \text{rank } T_j < \frac{1}{2}n + \frac{1}{2}n = n$ , which is impossible.

Let  $K = \ker(T - \alpha I) \cap \ker T_3$  and  $m = \dim \ker(T - \alpha I)$ . Then  $\dim K = \dim \ker(T - \alpha I) + \dim \ker T_3 - \dim[\ker(T - \alpha I) + \ker T_3] \geq m + \frac{1}{2}n - n = m - \frac{1}{2}n$ . Since  $K$  is invariant for both  $T$  and  $(I + T_3)^{-1}$ , it is invariant for  $(I + T_1)(I + T_2)$ . Moreover,  $(I + T_1)(I + T_2)|_K = T(I + T_3)^{-1}|_K = \alpha I_K$  ( $I_K$  denotes the identity matrix on  $K$ ). Hence  $K \subseteq \ker[(I + T_1)(I + T_2) - \alpha I]$ , which implies that  $\dim \ker[(I + T_1)(I + T_2) - \alpha I] \geq \dim K \geq m - \frac{1}{2}n$ . By Lemma 2.2,  $(I + T_1)(I + T_2)$  is similar to  $(I + T_2)^{-1}(I + T_1)^{-1}$ , whence we also have  $\dim \ker[(I + T_1)(I + T_2) - \alpha^{-1}I] \geq m - \frac{1}{2}n$ .

Let  $L = \ker(T - \alpha I) \cap \ker[(I + T_1)(I + T_2) - \alpha^{-1}I]$ . We repeat the above arguments:

$$\begin{aligned} \dim L &\geq \dim \ker(T - \alpha I) + \dim \ker[(I + T_1)(I + T_2) - \alpha^{-1}I] - n \\ &\geq m + (m - \frac{1}{2}n) - n \\ &= 2m - \frac{3}{2}n. \end{aligned} \quad (*)$$

Since  $L$  is invariant for both  $T$  and  $[(I + T_1)(I + T_2)]^{-1}$ , it is invariant for  $I + T_3$ . Moreover,  $(I + T_3)|_L = [(I + T_1)(I + T_2)]^{-1}T|_L = \alpha^2 I_L$ . Thus  $L \subseteq \ker[(1 - \alpha^2)I + T_3]$ . However, since  $T_3^2 = 0$ ,  $(1 - \alpha^2)I + T_3$  is invertible for any  $\alpha \in \mathbb{C}$ ,  $\alpha^2 \neq 1$ . Therefore  $\dim L = 0$ . From (\*), we infer that  $m \leq \frac{3}{4}n$ , as asserted. ■

Next we consider sufficient conditions for products of three unipotents. Our main tool is the following lemma.

**LEMMA 3.2.** *If  $T$  is an  $n \times n$  invertible cyclic matrix and  $\alpha_1, \dots, \alpha_n$  are complex numbers satisfying  $\alpha_1 \cdots \alpha_n = \det T$ , then there exist matrices  $A$  and  $B$  such that  $T = AB$ ,  $(A - I)^2 = 0$ , and  $B$  is cyclic with  $\sigma(B) = \{\alpha_1, \dots, \alpha_n\}$ .*

*Proof.* Since  $T$  is similar to a companion matrix of the form

$$C \equiv \begin{bmatrix} 0 & & & & & a_0 \\ 1 & 0 & & & & \vdots \\ & \ddots & \ddots & & & \vdots \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & \ddots & 0 \\ 0 & & & & 1 & a_{n-1} \end{bmatrix},$$

we need only prove our assertion for  $C$ . For  $j = 1, \dots, n-1$ , let  $b_j$  be the coefficient of  $x^j$  in the expansion of  $(x - \alpha_1) \cdots (x - \alpha_n)$ . Let

$$A = \begin{bmatrix} 1 & & & & \\ a_0^{-1}(a_1 + b_1) & 1 & & & \\ \vdots & & \ddots & & \\ a_0^{-1}(a_{n-1} + b_{n-1}) & & & 1 & \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & & & & a_0 \\ 1 & 0 & & & -b_1 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & 0 & \vdots \\ 0 & & & 1 & -b_{n-1} \end{bmatrix}$$

[noting that  $a_0 = (-1)^{n+1} \det T \neq 0$ ]. It is easily seen that  $C = AB$ ,  $(A - I)^2 = 0$ , and  $B$  is cyclic with characteristic polynomial  $(x - \alpha_1) \cdots (x - \alpha_n)$ . Hence  $\sigma(B) = \{\alpha_1, \dots, \alpha_n\}$ . ■

**PROPOSITION 3.3.** *Let  $T$  be a matrix with  $\det T = 1$ . If  $T = T_1 \oplus \cdots \oplus T_m$ , where each  $T_j$  is cyclic with size at least 2, then  $T$  is the product of three unipotents.*

*Proof.* Let  $d_j = \det T_j$ ,  $j = 1, \dots, m$ , and fix a nonzero number  $c$ . By Lemma 3.2, for each  $j$  there exist matrices  $A_j$  and  $B_j$  such that  $T_j = A_j B_j$ ,  $(A_j - I)^2 = 0$ , and  $\sigma(B_j) = \{c(\prod_{i=1}^{j-1} d_i)^{-1}, c^{-1}(\prod_{i=1}^j d_i), 1, \dots, 1\}$ . If  $A = A_1 \oplus \cdots \oplus A_m$  and  $B = B_1 \oplus \cdots \oplus B_m$ , then  $T = AB$ ,  $(A - I)^2 = 0$ , and  $\sigma(B) = \{b_1, \dots, b_{2m}, 1, \dots, 1\}$ , where the  $b_j$ 's satisfy  $b_j^{-1} = b_{j+m}$  for  $j = 1, \dots, m$ . Choose  $c$  such that the  $b_j$ 's are all distinct and different from  $\pm 1$ . By Theorem 2.7,  $B$  is the product of two unipotents, whence  $T$  is the product of three unipotents. ■

**COROLLARY 3.4.** *If  $T$  is such that  $\det T = 1$ ,  $-1 \notin \sigma(T)$ , and  $\dim \ker (T - \alpha I) \leq 2$  for any  $\alpha \neq 1$ , then  $T$  is the product of three unipotents.*

*Proof.* Using Proposition 3.3 and the rational form for matrices, we are reduced to considering  $T$  in the following form:

$$\begin{bmatrix} T_1 & 0 \\ 0 & a \end{bmatrix},$$

where  $a \neq 0, \pm 1$ , and  $T_1$  is cyclic with size at least 2 and characteristic

polynomial  $p_1$  satisfying  $p_1(a) = 0$ . Let  $A$  and  $B$  be such that  $T_1 = AB$ ,  $(A - I)^2 = 0$ , and  $B$  is cyclic with  $\sigma(B) = \{a^{-1}, 1, \dots, 1\}$ . Then  $B$  is similar to

$$\begin{bmatrix} a^{-1} & 0 \\ 0 & I + N \end{bmatrix},$$

where  $N$  is nilpotent. Hence

$$T = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & a \end{bmatrix}$$

with  $\begin{bmatrix} B & 0 \\ 0 & a \end{bmatrix}$  similar to

$$\begin{bmatrix} a^{-1} & 0 & 0 \\ 0 & I + N & 0 \\ 0 & 0 & a \end{bmatrix}.$$

This latter matrix is the product of two unipotents by Theorem 2.7. This proves our assertion for  $T$ . ■

We conclude this paper with the following theorem, which says that matrices with determinant 1 can always be written as the product of four unipotents.

**THEOREM 3.5.** *An  $n \times n$  matrix  $T$  is the product of finitely many unipotents if and only if  $\det T = 1$ . In this case, the minimal number of required unipotents is 1 if  $n = 1$ , 3 if  $n = 2$ , and 4 if  $n \geq 3$ .*

*Proof.* If  $\det T = 1$ , we want to show that  $T$  is the product of four unipotents. We consider the following two cases separately:

*Case 1.  $T$  is nonscalar.* By [11, Theorem 1], there exist matrices  $A$  and  $B$  such that  $T = AB$  and both  $\sigma(A)$  and  $\sigma(B)$  are of the form  $\{\alpha_1, \alpha_1^{-1}, \dots, \alpha_m, \alpha_m^{-1}\}$  or  $\{1, \alpha_1, \alpha_1^{-1}, \dots, \alpha_m, \alpha_m^{-1}\}$  according as  $n$  is even or odd, where the  $\alpha_i$ 's are chosen such that all the elements in  $\sigma(A)$  and  $\sigma(B)$  are distinct. Since both  $A$  and  $B$  are products of two unipotents by Theorem 2.7,  $T$  is the product of four unipotents.

*Case 2.  $T = \alpha I_n$  for some  $\alpha$  with  $\alpha^n = 1$ .* If  $n$  is odd, write  $T = AB$ , where  $A$  and  $B$  are diagonal matrices with diagonals  $\{\alpha, \alpha^{-1}, \alpha^3, \alpha^{-3}, \dots, \alpha^{n-2}, \alpha^{-(n-2)}, 1\}$  and  $\{1, \alpha^2, \alpha^{-2}, \dots, \alpha^{n-1}, \alpha^{-(n-1)}\}$ , respectively. Note that  $\alpha^j \neq -1$  for any  $j$ . Hence Theorem 2.7 implies that both  $A$  and  $B$  are

products of two unipotents. Similarly, if  $n$  is even, write  $T = AB$ , where  $A = \text{diag}(a^{-1}\alpha, a\alpha^{-1}, a^{-1}\alpha^3, a\alpha^{-3}, \dots, a^{-1}\alpha^{n-1}, a\alpha^{-(n-1)})$ ,  $B = \text{diag}(a, a^{-1}\alpha^2, a\alpha^{-2}, \dots, a^{-1}\alpha^{n-2}, a\alpha^{-(n-2)}, a^{-1})$  and  $a$  is any real number bigger than 1. Note that  $|a^{-1}\alpha^j| = a^{-1}$  implies that  $a^{-1}\alpha^j \neq -1$  for any  $j$ . Our assertion follows from Theorem 2.7.

If  $n \geq 3$ , then  $T = \alpha I_n$ , where  $\alpha^n = 1$  and  $\alpha \neq \pm 1$ , cannot be written as a product of three unipotents by Theorem 3.1. For  $n = 2$ , the assertion on the minimal number is a consequence of [4, Theorem 2]. ■

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